Aggregation Functions with Stronger Types of Monotonicity

Erich Peter Klement[1], Maddalena Manzi[2], and Radko Mesiar[3,4]

[1] Department of Knowledge-Based Mathematical Systems,
Johannes Kepler University, Linz, Austria
ep.klement@jku.at
[2] Department of Pure and Applied Mathematics,
University of Padua, Padova, Italy
mmanzi@math.unipd.it
[3] Department of Mathematics and Descriptive Geometry,
Faculty of Civil Engineering, Slovak University of Technology, Bratislava, Slovakia
mesiar@math.sk
[4] Institute of Theory of Information and Automation,
Czech Academy of Sciences, Prague, Czech Republic

Abstract. Following the ideas of stronger forms of monotonicity for unary real functions and for capacities, k-monotone and strongly k-monotone aggregation functions are introduced and discussed. In the special case $k = 2$ also some applications are given.

1 Introduction

The monotonicity of a real function $f\colon I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is some real interval, can be strengthened into the total monotonicity. Recall that a real function f is *totally monotone* if it is smooth and all its derivatives are nonnegative. In particular, a real function $f\colon [0,1] \to [0,1]$ is totally monotone if and only if $f(x) = \sum_{i=0}^{\infty} a_i \cdot x^i$ with $a_i \geq 0$ for all $i \in \mathbb{N} \cup \{0\}$ and $\sum_{i=0}^{\infty} a_i \leq 1$. Observe that if $f(0) = 0$ and $f(1) = 1$ are required then necessarily $a_0 = 0$ and $\sum_{i=0}^{\infty} a_i = 1$. Similarly, the monotonicity of capacities can be strengthened into the k-monotonicity, $k = 2, 3, \ldots, \infty$. Recall that, for a measurable space $(X, \mathcal{A})$, a mapping $m\colon \mathcal{A} \to [0,1]$ is called a *capacity* if $m(\emptyset) = 0$, $m(X) = 1$ and m is monotone, i.e., $m(E) \leq m(F)$ whenever $E \subseteq F$. For a fixed $k \in \mathbb{N} \setminus \{1\}$, m is called *k-monotone* if for all $E_1, \ldots, E_k \in \mathcal{A}$ we have

$$m\left(\bigcup_{i=1}^{k} E_i\right) \geq \sum_{\emptyset \neq J \subseteq \{1,\ldots,k\}} (-1)^{|J|+1} m\left(\bigcap_{j \in J} E_j\right) \qquad (1)$$

Moreover, if a capacity m satisfies (1) for all $k \in \mathbb{N} \setminus \{1\}$ then m is called an *∞-monotone capacity* (or, equivalently, a *belief measure*). For more details see [8, 10].

E. Hüllermeier, R. Kruse, and F. Hoffmann (Eds.): IPMU 2010, LNAI 6178, pp. 418–424, 2010.

The k-monotonicity (1) of a capacity m can be formulated in an equivalent way: m is k-monotone if for all $r \in \{2, \ldots, k\}$ and for all pairwise disjoint $E, E_1, \ldots, E_r \in \mathcal{A}$,

$$\sum_{J \subseteq \{1,\ldots,r\}} (-1)^{r-|J|} m\Big(E \cup \bigcup_{j \in J} E_j\Big) \geq 0. \tag{2}$$

Inequality (2) can be generalized to an arbitrary bounded lattice $(L, \vee, \wedge, \mathbf{0}, \mathbf{1})$. Indeed, let $g\colon L \to \mathbb{R}$ be a non-decreasing mapping, i.e., $g(a) \leq g(b)$ whenever $a \leq b$. Then g is *k-monotone*, $k \in \mathbb{N} \setminus \{1\}$, if for all $r \in \{2, \ldots, k\}$, for all $a \in L$, and for all pairwise disjoint $a_1, \ldots, a_r \in L$ (i.e., $a_1 \wedge a_2 = \mathbf{0}$, etc.) we have

$$\sum_{J \subseteq \{1,\ldots,r\}} (-1)^{r-|J|} g\Big(a \vee \bigvee_{j \in J} a_j\Big) \geq 0. \tag{3}$$

If the lattice L under consideration is a sublattice of some vector lattice (and if $\mathbf{0}$ is the neutral element of the addition on that vector space) then another condition equivalent to (3) can be given: a non-decreasing mapping $g\colon L \to \mathbb{R}$ is k-monotone if for all $r \in \{2, \ldots, k\}$ and for all $a, a_1, \ldots, a_r \in L$ with

$$a = a + \bigvee a_i = a + a_1 + \cdots + a_r \in L$$

we have

$$\sum_{J \subseteq \{1,\ldots,r\}} (-1)^{r-|J|} g\Big(a + \sum_{j \in J} a_j\Big) \geq 0. \tag{4}$$

(observe that $\bigvee a_i = a_1 + \cdots + a_r$ is equivalent to $a_1, \ldots, a_r$ being pairwise disjoint).

Moreover, in this case the following strong k-monotonicity related to (4) can be introduced: a non-decreasing mapping $g\colon L \to \mathbb{R}$ is called *strongly k-monotone* if for all $r \in \{2, \ldots, k\}$ and for all $a, a_1, \ldots, a_r \in L$ with $a + a_1 + \cdots + a_r \in L$ we have

$$\sum_{J \subseteq \{1,\ldots,r\}} (-1)^{r-|J|} g\Big(a + \sum_{j \in J} a_j\Big) \geq 0. \tag{5}$$

Observe that if $(L, \vee, \wedge, \mathbf{0}, \mathbf{1}) = (\mathcal{A}, \cup, \cap, \emptyset, X)$ then conditions (2) and (3) coincide (if we put $m = g$). Moreover, taking into account that each set $E \in \mathcal{A}$ is represented by the corresponding characteristic function $\mathbf{1}_E$, then $\iota\colon \mathcal{A} \to \mathbb{R}^X$ defined by $\iota(E) = \mathbf{1}_E$ provides an embedding of $(\mathcal{A}, \cup, \cap, \emptyset, X)$ into the vector lattice $(\mathbb{R}^X, \sup, \inf, \mathbf{0}, \mathbf{1})$, where $\mathbf{0}$ and $\mathbf{1}$ are the constant functions assuming only the value 0 and 1, respectively. Then $\iota(\mathcal{A})$ is a bounded sublattice of $\mathbb{R}^X$ (and even a sublattice of $\{0,1\}^X$). Putting $g(\mathbf{1}_E) = m(E)$, we see the equivalence of (2), (4) and (5).

This contribution aims at discussing aggregation functions $A\colon [0,1]^n \to [0,1]$ which are k-monotone or strongly k-monotone. As more details on aggregation functions can be found in the recent monograph [3], here we only recall that, for a fixed $n \in \mathbb{N}$, a real function $A\colon [0,1]^n \to [0,1]$ is called an *aggregation function* if it is non-decreasing and satisfies $A(0, \ldots, 0) = 0$ and $A(1, \ldots, 1) = 1$.

The paper is organized as follows. In Section 2, k-monotone and strongly k-monotone aggregation functions are discussed in general, i.e., for $k = 2, 3, \ldots, \infty$. Under some specific requirements, well-known aggregation functions are recovered. Section 3 is devoted to the particular cases $k = 2$ and $k = \infty$, while in Section 4 some possible applications are indicated. Finally, several open problems are posed.

2 (Strongly) k-Monotone Aggregation Functions

Based on (4) and (5), we introduce the following stronger forms of monotonicity for aggregation functions.

Definition 1. Let $A\colon [0,1]^n \to [0,1]$ be an aggregation function and $k \in \mathbb{N}\setminus\{1\}$.

(i) The aggregation function A is called *k-monotone* if for each $r \in \{2, \ldots, k\}$ and for all $\mathbf{x}, \mathbf{x}_1, \ldots, \mathbf{x}_r \in [0,1]^n$ with $\mathbf{x} + \mathbf{x}_1 + \cdots + \mathbf{x}_r = \mathbf{x} + \bigvee \mathbf{x}_i \in [0,1]^n$ we have

$$\sum_{J \subseteq \{1,\ldots,r\}} (-1)^{r-|J|} A\Big(\mathbf{x} + \bigvee_{j \in J} \mathbf{x}_j\Big) \geq 0. \tag{6}$$

(ii) The aggregation function A is said to be *strongly k-monotone* if for each $r \in \{2, \ldots, k\}$ and for all $\mathbf{x}, \mathbf{x}_1, \ldots, \mathbf{x}_r \in [0,1]^n$ with $\mathbf{x}+\mathbf{x}_1+\cdots+\mathbf{x}_r \in [0,1]^n$ we have

$$\sum_{J \subseteq \{1,\ldots,r\}} (-1)^{r-|J|} A\Big(\mathbf{x} + \sum_{j \in J} \mathbf{x}_j\Big) \geq 0. \tag{7}$$

(iii) The aggregation function A is called *strongly ∞-monotone* (*totally monotone*) if it is strongly k-monotone for each $k \in \mathbb{N} \setminus \{1\}$.

Note that if $\mathbf{x}+\mathbf{x}_1+\cdots+\mathbf{x}_r = \mathbf{x}+\bigvee \mathbf{x}_i \in [0,1]^n$ then formulae (6) and (7) coincide (and then $\mathbf{x}_1, \ldots, \mathbf{x}_r$ have pairwise disjoint supports, i.e., $\min(\mathbf{x}_i, \mathbf{x}_j) = \mathbf{0}$ for all $i \neq j$). Clearly, for an n-ary aggregation function A, its k-monotonicity for $k > n$ is equivalent to the n-monotonicity of A, which is not true for strong monotonicity. For example, for a unary aggregation function $f\colon [0,1] \to [0,1]$, k-monotonicity is just the non-decreasingness of f, while strong 2-monotonicity of f is equivalent to its convexity.

The following results can be found in [1].

Proposition 1. *Let $f\colon [0,1] \to [0,1]$ be an aggregation function. Then we have:*

(i) *f is strongly k-monotone for some $k \in \mathbb{N} \setminus \{1\}$ if and only if all derivatives of f of order $1, \ldots, k-2$ are nonnegative and $f^{(k-2)}$ is a non-decreasing convex function.*
(ii) *f is strongly ∞-monotone if and only if f is a totally monotone real function, i.e., it has non-negative derivatives of all orders on $[0,1[$.*

Proposition 2. *Let* $A\colon [0,1]^n \to [0,1]$ *be an aggregation function. Then* A *is totally monotone if and only if all partial derivatives of* A *are nonnegative. In particular, this means that*

$$A(u_1,\dots,u_n) = \sum a_{i_1,\dots,i_n} \cdot {u_1}^{i_1} \cdots {u_n}^{i_n},$$

where $i_1,\dots,i_n$ *run from* 0 *to* ∞, $a_{0,\dots,0} = 0$, *all* $a_{i_1,\dots,i_n} \geq 0$, *and* $\sum a_{i_1,\dots,i_n} = 1$.

As a particular consequence of Proposition 2 we see that, for each $n \in \mathbb{N}$, the product $\Pi\colon [0,1]^n \to [0,1]$ is a totally monotone aggregation function. Also, each weighted arithmetic mean $W\colon [0,1]^n \to [0,1]$ given by $W(u_1,\dots,u_n) = \sum w_i \cdot u_i$ is totally monotone.

Proposition 3. *Fix* $k \in \{2,3,\dots,\infty\}$. *Then for all* $n,m \in \mathbb{N}$ *and for all strongly* k*-monotone* n*-ary aggregation functions* $A\colon [0,1]^n \to [0,1]$ *and for all strongly* k*-monotone* m*-ary aggregation functions* $B_1,\dots,B_n\colon [0,1]^m \to [0,1]$ *also the composite function* $D\colon [0,1]^m \to [0,1]$ *given by*

$$D(\mathbf{x}) = A(B_1(\mathbf{x}),\dots,B_n(\mathbf{x}))$$

is strongly k*-monotone.*

It is possible to show that for each fixed $n \in \mathbb{N}$ and $k \in \{2,3,\dots,\infty\}$, the class of all (strongly) k-monotone n-ary aggregation functions is convex and compact (with respect to the topology of pointwise convergence).

For $n \in \mathbb{N} \setminus \{1\}$ and for n-ary aggregation functions $A\colon [0,1]^n \to [0,1]$, the notion of n-increasingness was introduced in the framework of copulas [7, 9]:

Definition 2. Let $n \geq 2$. An aggregation function $A\colon [0,1]^n \to [0,1]$ is called *n-increasing* if for all $\mathbf{x},\mathbf{y} \in [0,1]^n$ with $\mathbf{x} \leq \mathbf{y}$ we have

$$\sum_{J \subseteq \{1,\dots,n\}} (-1)^{n-|J|} A(\mathbf{z}_J) \geq 0, \tag{8}$$

where $\mathbf{z}_J \in [0,1]^n$ is given by $z_j = y_j$ if $j \in J$, and $z_j = x_j$ otherwise.

It is not difficult to check that, under the hypotheses of Definition 2, formulae (8) and (6) coincide, i.e., n-monotonicity and n-increasingness for n-ary aggregation functions mean the same. Hence, k-monotonicity extends the notion of n-increasingness to higher dimensions.

Remark 1

(i) Because of [1], strong k-monotone aggregation functions are important in the theory of non-additive measures: for k-monotone capacities $m_1,\dots,m_n$ acting on a fixed measurable space $(X,\mathcal{A})$ and for a strongly k-monotone n-ary aggregation function A, the set function $A(m_1,\dots,m_n)\colon \mathcal{A} \to [0,1]$ given by

$$A(m_1,\dots,m_n)(E) = A(m_1(E),\dots,m_n(E))$$

is a k-monotone capacity whenever A is strongly k-monotone (if $|X| \geq k$, this is also necessary condition if the claim should be valid for arbitrary k-monotone capacities $m_1,\dots,m_n$).

(ii) k-monotonicity is an axiom for k-dimensional copulas [9].
(iii) Strong 2-monotonicity is known also as *ultramodularity*, and it was discussed in general in [6] (see also [4]). Another name for 2-monotonicity is *supermodularity*, a widely used concept in the theory of non-additive measures and of aggregation functions.

3 (Strongly) 2-Monotone Aggregation Functions

Recall that an aggregation function $C\colon [0,1]^2 \to [0,1]$ which is 2-monotone and satisfies $C(x,1) = C(1,x) = x$ for all $x \in [0,1]$ is called a 2-*copula* (or, shortly, a *copula*). Copulas play a key role in the description of the stochastic dependence of two-dimensional random vectors and they are substantially exploited in several applications in finance, hydrology, etc. The construction of new types of copulas is one of the important theoretical tasks allowing a better modelling of real problems involving stochastic uncertainty. From [2] we have the following representation result:

Proposition 4. *An aggregation function $A\colon [0,1]^2 \to [0,1]$ is 2-monotone if and only if there are non-decreasing functions $g_1, g_2, g_3, g_4\colon [0,1] \to [0,1]$ with $g_i(0) = 0$ and $g_i(1) = 1$ for each $i \in \{1,2,3,4\}$, a binary copula $C\colon [0,1]^2 \to [0,1]$, and numbers $a,b,c \in [0,1]$ with $a+b+c = 1$ such that, for all $(x,y) \in [0,1]^2$,*

$$A(x,y) = a \cdot g_1(x) + b \cdot g_2(y) + c \cdot C(g_3(x), g_4(y)). \tag{9}$$

If 0 is an annihilator of the aggregation function $A\colon [0,1]^2 \to [0,1]$, i.e., if $A(x,0) = A(0,x) = 0$ for all $x \in [0,1]$, then (9) reduces to

$$A(x,y) = C(f(x), g(y)), \tag{10}$$

where $f, g\colon [0,1] \to [0,1]$ are non-decreasing functions with $f(0) = g(0) = 0$ and $f(1) = g(1) = 1$. Note that then we have $f(x) = A(x,1)$ and $g(x) = A(1,x)$ for all $x \in [0,1]$. The following result can be derived from [6].

Proposition 5. *An aggregation function $A\colon [0,1]^2 \to [0,1]$ is strongly 2-monotone if and only if A is 2-monotone and each horizontal and each vertical section of A is a convex function.*

In the class of copulas, the greatest strongly 2-monotone copula is the product copula Π, while the smallest strongly 2-monotone copula is the Fréchet-Hoeffding lower bound W given by $W(x,y) = \max(x+y-1, 0)$. Note that the only totally monotone 2-copula is the product copula Π. The following theorem will be helpful in the construction of copulas.

Theorem 1. *Let $A\colon [0,1]^n \to [0,1]$ be an aggregation function and $k \geq 2$. Then the following are equivalent:*

(i) *A is strongly 2-monotone.*
(ii) *If $B_1, \ldots, B_n\colon [0,1]^k \to [0,1]$ are non-decreasing 2-monotone functions then the composite $D\colon [0,1]^k \to [0,1]$ given by $D(\mathbf{x}) = A(B_1(\mathbf{x}), \ldots, B_n(\mathbf{x}))$ is a 2-monotone function.*

4 Construction of Copulas

Theorem 2. *Let* $A\colon [0,1]^n \to [0,1]$ *be a continuous, strongly* 2*-monotone aggregation function. Let* $C_1, \ldots, C_n\colon [0,1]^2 \to [0,1]$ *be copulas and assume that the functions* $f_1, \ldots, f_n, g_1, \ldots, g_n\colon [0,1] \to [0,1]$ *satisfy* $f_i(1) = g_i(1) = 1$ *for each* $i \in \{1, \ldots, n\}$ *and* $A(f_1(0), \ldots, f_n(0)) = A(g_1(0), \ldots, g_n(0)) = 0$*. Define* $\xi, \eta\colon [0,1] \to [0,1]$ *by*

$$\xi(x) = \sup\{u \in [0,1] \mid A(f_1(u), \ldots, f_n(u)) \le x\},$$
$$\eta(x) = \sup\{u \in [0,1] \mid A(g_1(u), \ldots, g_n(u)) \le x\}.$$

Then the function $C\colon [0,1]^2 \to [0,1]$ *given by*

$$C(x, y) = A\big(C_1(f_1 \circ \xi(x), g_1 \circ \eta(y)), \ldots, C_n(f_n \circ \xi(x), g_n \circ \eta(y))\big) \tag{11}$$

is a copula.

For k-monotone aggregation functions we have the following result.

Theorem 3. *Let* $A\colon [0,1]^n \to [0,1]$ *be a totally monotone aggregation function, and let* $B_1, \ldots, B_n\colon [0,1]^m \to [0,1]$ *be* k*-monotone aggregation functions. Then the composite function* $D\colon [0,1]^m \to [0,1]$ *given by* $D(\mathbf{x}) = A(B_1(\mathbf{x}), \ldots, B_n(\mathbf{x}))$ *is a* k*-monotone aggregation function.*

This result can be applied to the construction of k-dimensional copulas (i.e., k-monotone aggregation functions $C\colon [0,1]^k \to [0,1]$ satisfying

$$C(x, 1, \ldots, 1) = C(1, x, \ldots, 1) = C(1, \ldots, 1, x) = x$$

for all $x \in [0,1]$) in a way similar to Theorem 2.

Example 1

(i) If we put $n = 2$, $A = W$, $C_1 = C_2 = M$ and define the functions f_1, f_2, g_1, g_2 by $f_1(x) = g_2(x) = \frac{x+2}{3}$ and $f_2(x) = g_1(x) = \frac{2x+1}{3}$, then the construction in (11) yields the copula C given by

$$C(x, y) = \frac{1}{3} \cdot \max(\min(x + 1, 2y) + \min(2x, y + 1) - 1, 0).$$

(ii) Consider the totally monotone aggregation function $A\colon [0,1]^n \to [0,1]$ given by $A(\mathbf{x}) = x_1^{p_1} \cdots x_n^{p_n}$, where $p_1, \ldots, p_n \in \mathbb{N} \cup \{0\}$ and $p = \sum p_i > 0$. Then for all k-dimensional copulas $C_1, \ldots, C_n\colon [0,1]^k \to [0,1]$, the aggregation function $C\colon [0,1]^k \to [0,1]$ given by $C(\mathbf{x}) = A(C_1(\tau(\mathbf{x})), \ldots, C_n(\tau(\mathbf{x})))$, where $\tau\colon [0,1]^k \to [0,1]^k$ is given by $\tau(\mathbf{x}) = (x_1^{1/p}, \ldots, x_k^{1/p})$, is a k-dimensional copula. This result can be derived also from [5]. For example, for $n = 2$ and $A(x, y) = x \cdot y^2$ (i.e., $p = 3$) and for the ternary copulas $C_1 = M$ (i.e., $M(x, y, z) = \min(x, y, z)$) and $C_2 = \Pi$ (i.e., $\Pi(x, y, z) = xyz$), the composite function $C\colon [0,1]^3 \to [0,1]$ given by

$$C(x, y, z) = \min\Big(x(yz)^{\frac{2}{3}}, y(xz)^{\frac{2}{3}}, z(xy)^{\frac{2}{3}}\Big)$$

is a ternary copula.

5 Concluding Remarks

We have introduced two properties which are stronger than the monotonicity of aggregation functions, with some representation results and with an application for constructing copulas. Our proposal opens several new questions for future research. For example, it is not clear whether there are strongly 3-monotone copulas different from the product Π. Also it is still open whether/how the conditions of Theorem 3 can be relaxed yielding still the same result — is the strong k-monotonicity of A sufficient? We also expect applications in the construction of copulas of higher dimensions, and subsequently, in the modeling of stochastic dependence of random vectors with dimension $n \geq 3$ (note that, so far, there are only few methods in this case known in the literature).

Acknowledgment

The first and the third author were supported by a bilateral project Austria-Slovakia (WTZ, Project SK 04/2009), the second and the third author by the grant APVV-0012-07, and the third author also by the grant VEGA 1/0080/10.

References

1. Bronevich, A.G.: On the closure of families of fuzzy measures under eventwise aggregations. Fuzzy Sets and Systems 153, 45–70 (2005)
2. Durante, F., Saminger-Platz, S., Sarkoci, P.: On representations of 2-increasing binary aggregation functions. Inform. Sci. 178, 4534–4541 (2008)
3. Grabisch, M., Marichal, J.-L., Mesiar, R., Pap, E.: Aggregation Functions. Cambridge University Press, Cambridge (2009)
4. Klement, E.P., Manzi, M., Mesiar, R.: Ultramodular aggregation functions and a new construction method for copulas (submitted for publication)
5. Liebscher, E.: Construction of asymmetric multivariate copulas. J. Multivariate Anal. 99, 2234–2250 (2008)
6. Marinacci, M., Montrucchio, L.: Ultramodular functions. Math. Oper. Res. 30, 311–332 (2005)
7. Nelsen, R.B.: An Introduction to Copulas, 2nd edn. Lecture Notes in Statistics, vol. 139. Springer, New York (2006)
8. Pap, E.: Null-Additive Set Functions. Kluwer Academic Publishers, Dordrecht (1995)
9. Sklar, A.: Fonctions de répartition à n dimensions et leurs marges. Publ. Inst. Statist. Univ. Paris 8, 229–231 (1959)
10. Wang, Z., Klir, G.J.: Generalized Measure Theory. Springer, New York (2009)